

HIGHER ORDER COMMUTATORS FOR VECTOR-VALUED CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. Weighted norm estimates for higher order commutators are obtained. The proof, that remain valid in the vector-valued case, are obtained as an application of some extrapolation results. The vector-valued version of the commutator theorem is applied to the Carleson operator, U.M.D. Banach spaces, approximate identities and maximal operators.

INTRODUCTION

The purpose of this paper is to obtain weighted norm estimates for higher order commutators, see Theorem (2.1). The proof of this commutator theorem is an application of some extrapolation results of independent interest stated in §1. We give proofs that remain valid in the vector-valued case. We apply the vector-valued version of the commutator theorem to the Carleson operator, U.M.D. Banach spaces, approximate identities and maximal operators.

Throughout this paper we shall work on \mathbf{R}^n endowed with the Lebesgue measure. Given a Banach space E we shall denote by $L_E^p(\mathbf{R}^n)$ or simply L_E^p the Bochner-Lebesgue spaces of E -valued strongly measurable functions such that $\int_{\mathbf{R}^n} \|f(x)\|_E^p dx < \infty$.

Given a nonnegative measurable function $\alpha(x)$, we shall denote by $L_E^p(\alpha)$ the space of E -valued strongly measurable functions such that

$$\int_{\mathbf{R}^n} \|f(x)\|_E^p \alpha(x) dx < \infty.$$

By $\text{BMO}_E(\nu)$ we mean the space of E -valued strongly measurable functions $a(x)$ such that $\int_Q \|a(x) - a_Q\|_E dx \leq C \int_Q \nu(x) dx$ holds for every ball Q and $a_Q = |Q|^{-1} \int_Q a(x) dx$ with a constant C independent of Q .

Given two Banach spaces E and F , we denote $\mathcal{L}(E, F)$ the Banach space of all continuous linear operators from E into F .

By a Banach lattice we mean an ordered Banach space F over the real numbers satisfying

(i) for every x and y in F there exist a least upper bound $x \vee y$, and a greatest lower bound $x \wedge y$.

(ii) if $|x| = x \vee (-x)$ then $\|x\| \leq \|y\|$, whenever $|x| \leq |y|$.

We observe that $\|x\| = \|(|x|)\|$ is a consequence of (ii).

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Let $\alpha(x)$ be a real valued and nonnegative function defined on \mathbf{R}^n . We shall say that α belongs to the class $A(p, q)$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, if

$$(0.1) \quad \left(|Q|^{-1} \int_Q \alpha(x)^{-p'} dx \right)^{1/p'} \left(|Q|^{-1} \int_Q \alpha(x)^q dx \right)^{1/q} \leq c$$

holds for every ball Q with a constant c independent of Q . Here $p' + p = p \cdot p'$. The least constant c satisfying (0.1) shall be called the constant of α in the class $A(p, q)$. The Hardy-Littlewood maximal function $M(f)(x)$ is defined as $\sup\{|f|_Q : x \in Q\}$. It is well known that for $1 < p \leq \infty$,

$$\int |Mf(x)|^p \omega(x)^p dx \leq c_\omega \int |f(x)|^p \omega(x)^p dx$$

holds if and only if $\omega \in A(p, p)$, see [Mu and RFC]. The “sharp” function of a function f is defined as

$$f^\#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q \|f(x) - f_Q\| dx.$$

As a reference for properties of the maximal function, the sharp function and weights see [RFC].

1. THE EXTRAPOLATION THEOREM (THE INFINITY CASE)

We begin this section with a lemma due to Rubio de Francia [RFC, p. 434].

(1.1) **Lemma.** *Let S be an operator, $\rho(x) \geq 0$ and a number $t \geq 1$. Let us assume that*

- (i) $S(f) \geq 0$,
- (ii) $S(f_1 + f_2) \leq S(f_1) + S(f_2)$,
- (iii) $S(\lambda f) = |\lambda|S(f)$, and
- (iv) $\|\rho S(f)\|_t \leq c\|\rho f\|_t$.

Then, if ρu belongs to L^t , there exists $U \geq u$ such that

$$\|\rho U\|_t \leq 2\|\rho u\|_t \quad \text{and} \quad S(U) \leq 2cU.$$

As an application of Rubio de Francia's Lemma, we obtain

(1.2) **Proposition.** *Let $\mu(x)$ and $\eta(x)$ be nonnegative functions. Let $v \geq 1$ and $1 < s < \infty$. We assume that, μ and $\eta\mu$ belong to $A(s, st)$ where $s' > v$ and $st = s'v(s' - v)$. Then if $h(x) \geq 0$ and $h\mu^{s/s'}$ belongs to $L^{s'}$, there exists $H \geq h$ such that $\|H\mu^{s/s'}\|_{s'} \leq 2\|h\mu^{s/s'}\|_{s'}$ and both, $\mu^s H$ and $\eta\mu^s H$, belong to $A(1, v)$.*

Proof. Let us define the operator S as

$$S(f) = \mu^{-s}[M((\mu^s f)^v)]^{1/v} + \eta^{-1}\mu^{-s}[M((\eta\mu^s f))]^{1/v}.$$

This operator S satisfies conditions (i) to (iv) of Lemma (1.1) with $\rho = \mu^{s/s'}$ and $t = s'$. Then, there exists a function $H \geq h$ such that

$$\mu^{-s}[M((\mu^s H)^v)]^{1/v} \leq S(H) \leq 2cH$$

and

$$\eta^{-1}\mu^{-s}[M((\eta\mu^s H)^v)]^{1/v} \leq S(H) \leq 2cH,$$

showing that $\mu^s H$ and $\eta\mu^s H$ belong to $A(1, v)$.

Now, we state and prove the main result of this section. For previous versions of this theorem see [HMS and STa].

(1.3) **Theorem** (extrapolation theorem, the infinite case). *Let T be a sublinear operator defined on C_0^∞ , $1 \leq v < \infty$, $1 < r < \infty$, $r < p < \lambda \leq \infty$, $v < (q/r)/(\lambda/r)'$, $q^{-1} = p^{-1} - \lambda^{-1}$. Let $\nu(x) \geq 0$ and assume that*

$$(1.4) \quad \|\omega T f\|_\infty \leq c_\omega \|\omega f\|_\lambda$$

holds for every ω such that ω and $\nu\omega$ belong to $A(((\lambda/r)'v)', \infty)$ with a constant c_ω depending on the constants of ω and $\nu\omega$ in this class. Then, if ω^r and $(\nu\omega)^r$ belong to $A((p/r)'t)', q/r)$, where t is given by $1/(q/r) = 1/\nu(\lambda/r)' - 1/t(p/r)'$, we have that

$$(1.5) \quad \|\omega T f\|_q \leq c_\omega \|\omega f\|_p,$$

hold for every $f \in C_0^\infty$ with c_ω a constant depending on the constants of ω^r and $(\nu\omega)^r$ in their class only.

Moreover, if the hypotheses of the theorem holds for every v , $1 < v < (q/r)/(\lambda/r)'$, then if ω^r and $(\nu\omega)^r$ belong to $A(p/r, q/r)$ it follows that (1.5) holds for this ω , since there exists $1 < v$ close enough to 1 such that $\omega \in A(((p/r)'t)', q/r)$.

Proof. We can assume without loss of generality that $\|\omega f\|_p = 1$. Let h be the function defined as

$$h = \{|f| \omega^{(p/r)'}\}^{r(\lambda/r)'p/q}.$$

This function satisfies

$$(1.6) \quad \int h^{(q/r)/(\lambda/r)'} \omega^{-r(\lambda/r)'(1+(p/r)'/(q/r))} dx = 1$$

and

$$(1.7) \quad \left(\int |f h^{-1/r(\lambda/r)'} \omega^{(p/r)'/(q/r)'}|^\lambda dx \right)^{1/\lambda} = 1.$$

Let $\eta = \nu^{-r(\lambda/r)'}$, $\mu = \omega^{-r(\lambda/r)'}$ and $s = 1 + (p/r)'/(q/r)$.

With these notations we have $s' = s/(s-1) = (q/r)/(\lambda/r)'$, $st = s'v/(s'-v)$ and $\mu \in A(s, st)$. Moreover, (1.6) becomes

$$\left(\int h^{s'} \mu^s dx \right)^{1/s'} = \|h \mu^{s/s'}\|_{s'} = 1.$$

Then, by Proposition (1.2) there exists $H \geq h$ such that

$$(1.8) \quad \left(\int H^{s'} \mu^s \right)^{1/s'} \leq 2,$$

and

$$(1.9) \quad \text{both } \mu^s H \text{ and } \eta \mu^s H \text{ belong to } A(1, v).$$

Let $G = H^{-1/r(\lambda/r)'} \omega^{(p/r)'/(q/r)'}$. We shall prove that G^r and $(\nu G)^r$ belong to $A(((\lambda/r)'v)', \infty)$ or, equivalently, that $G^{-r(\lambda/r)'}$ and $(\nu G)^{-r(\lambda/r)'}$ belong to

$A(1, v)$. In fact, since $r(\lambda/r)'s = r(p/r)'$, (1.9) tells us that

$$(1.10) \quad G^{-r(\lambda/r)'} = H\omega^{-r(p/r)'} = \mu^s H \in A(1, v)$$

and

$$(1.11) \quad (\nu G)^{-r(\lambda/r)'} = H\nu^{-r(\lambda/r)'} \omega^{-r(p/r)'} = \eta\mu^s H \in A(1, v).$$

Now, recalling (1.7), we get

$$1 \geq \left(\int |fH^{-1/r(\lambda/r)'} \omega^{(p/r)' / (\lambda/r)'}|^\lambda dx \right)^{1/\lambda} = \left(\int |fG|^\lambda \right)^{1/\lambda}.$$

Then, by (1.10), (1.11), (1.8) and (1.4), it follows that

$$\begin{aligned} 1 &\geq c_\omega^{-1} \|GTf\|_\infty c \left(\int H^{s'} \mu^s dx \right)^{1/q} \\ &\geq c \left(\int G^q H^{s'} \mu^s |Tf|^q dx \right)^{1/q} \geq c \left(\int \omega^q |Tf|^q dx \right)^{1/q}, \end{aligned}$$

as we wanted to show.

2. COMMUTATORS FOR FRACTIONAL AND SINGULAR INTEGRALS

In this section we state the main results of this paper.

(2.1) **Theorem** (Commutator theorem). *Let E and F be Banach spaces and K a bounded linear operator from $L_E^p(R^n)$ into $L_F^q(R^n)$, $q^{-1} = p^{-1} - \lambda^{-1}$, $1 < p < \lambda$. Let us assume that there exists a kernel $k(x, y)$ with values in $\mathcal{L}(E, F)$, such that*

(2.2) *if $f \in C_0^\infty$, then*

$$Kf(x) = \int k(x, y)f(y) dy \quad \text{if } x \notin \text{supp } f,$$

(2.3) *there exists a slowly increasing function $\omega(t) \geq 0$ ($\omega(2t) \leq c\omega(t)$) such that*

$$\begin{aligned} &\left(\int_{R < |y-y_0| < 2R} \|k(x, y) - k(z, y)\|^{r'} dy \right)^{1/r'} \\ &\leq cR^{-n/r(\lambda/r)'} \left(|x - z|/R + \int_{|x-z|/2R < t < |x-z|/R} \omega(t) dt/t \right). \end{aligned}$$

Let us assume that two functions $a(x)$ and $\tilde{a}(x)$ defined on R^n and with values in $\mathcal{L}(E, E)$ and $\mathcal{L}(F, F)$, respectively, are given and satisfy

(i) *for every x and y in R^n ,*

$$a(x)a(y) = a(y)a(x) \quad \text{and} \quad \tilde{a}(x)\tilde{a}(y) = \tilde{a}(y)\tilde{a}(x)$$

hold,

(ii) *for every $z \in R^n$,*

$$\tilde{a}(z)K(f)(x) = K(a(z)f)(x)$$

holds, and

(iii) a belongs to $\mathbf{BMO}_{\mathcal{L}(E,E)}(\nu)$ and \tilde{a} belongs to $\mathbf{BMO}_{\mathcal{L}(F,F)}(\nu)$.
We define the m -commutator $C_{a,\tilde{a}}^m(f)$ as

$$\begin{aligned} C_{a,\tilde{a}}^0(f)(x) &= Kf(x) \text{ and, if } m > 0, \\ C_{a,\tilde{a}}^m(f)(x) &= \tilde{a}(x)C_{a,\tilde{a}}^{m-1}(f)(x) - C_{a,\tilde{a}}^{m-1}(af)(x). \end{aligned}$$

Then, given α and β such that α^r and β^r belong to $A(p/r, q/r)$, $1 \leq r < p < \lambda \leq \infty$, $q^{-1} = p^{-1} - \lambda^{-1}$ and $\alpha\beta^{-1} = \nu^m$ there exists a finite constant c such that

$$\left(\int \|C_{a,\tilde{a}}^m(f)(x)\|_F^q \beta(x)^q dx \right)^{1/q} \leq c \left(\int \|f\|_E^p \alpha(x)^p dx \right)^{1/p}$$

holds provided that the Dini condition

$$\int_0^1 \lg^m(1/t) \omega(t) dt/t < \infty,$$

is satisfied.

For the case $m = 0$ this theorem is essentially due to Kurtz and Wheeden, see [KW]. For any $m \geq 1$ and $\nu = 1$, see [CRW]. For $m = 1$ and the Hilbert transform, see [B].

(2.4) **Theorem.** Let F be a Banach lattice and $W(f)$ a bounded linear operator from $L^p(R^n)$ into $L_F^q(R^n)$, $1 < p < \lambda \leq \infty$, $q^{-1} = p^{-1} - \lambda^{-1}$. Let us assume the existence of an F -valued kernel $w(x, y)$ such that

(W₁) for every $f \in C_0^\infty$,

$$W(f)(x) = \int w(x, y)f(y) dy$$

holds,

(W₂) for every x and y in R^n , $w(x, y) \geq 0$, and

(W₃) the kernel $w(x, y)$ satisfies condition (2.3), substituting $k(x, y)$ for $w(x, y)$.

Let us assume that a real valued function $a(x)$ is given and belongs to $\mathbf{BMO}(\nu)$. Then, the operator

$$W_a^m(f)(x) = \int w(x, y)|a(x) - a(y)|^m f(y) dy,$$

is well defined on C_0^∞ and there exists a finite constant c such that

$$\left(\int \|W_a^m(f)(x)\|_F^q \beta(x)^q dx \right)^{1/q} \leq c \left(\int |f(y)|^p \alpha(x)^p dx \right)^{1/p}$$

holds whenever α^r and β^r belong to $A(p/r, q/r)$, $1 \leq r < p < \lambda \leq \infty$, $q^{-1} = p^{-1} - \lambda^{-1}$, and $\alpha\beta^{-1} = \nu^m$, provided that

$$\int_0^1 \lg^m(1/t) \omega(t) dt/t < \infty.$$

3. APPLICATIONS

A. Singular integrals and fractional integrals. Let $\Omega(x)$ be a homogeneous function of degree zero defined on $R^n - \{0\}$. We assume that $\int_{|x'|=1} \Omega(x') dx' = 0$ and if ρ is a rotation in R^n and $|\rho| = \|\rho - I\|$, let

$$\sup_{|\rho| < t} \left(\int_{|x'|=1} |\Omega(\rho x') - \Omega(x')|^{r'} dx' \right)^{1/r'} = \omega(t), \quad 1 \leq r < \infty.$$

We assume that $\omega(t)$ satisfies the condition

$$(3.1) \quad \int_0^1 \lg^m(1/t) \omega(t) dt/t < \infty.$$

Then, by [KW] we obtain that the kernel $k(x-y) = \Omega(x-y)|x-y|^{-n/\lambda'}$, $1 < \lambda \leq \infty$, satisfies (2.3) and

$$Kf(x) = \text{p.v.} \int \frac{\Omega(x-y)}{|x-y|^{n/\lambda'}} \cdot f(y) dy,$$

is a bounded operator from L^p into L^q for every $1 < p < \lambda$, $q^{-1} = p^{-1} - \lambda^{-1}$. Thus, by the commutator theorem (2.1) we obtain that if $a \in \text{BMO}(\nu)$, then

$$C_a^m(f)(x) = \text{p.v.} \int \frac{\Omega(x-y)}{|x-y|^{n/\lambda'}} (a(x) - a(y))^m f(y) dy,$$

satisfies

$$\|\beta C_a^m f\|_q \leq c \|\alpha f\|_p,$$

for every α and β , $\alpha\beta^{-1} = \nu^m$, α^r and β^r in $A(p/r, q/r)$ and $q^{-1} = p^{-1} - \lambda^{-1}$, $1 \leq r < p < \lambda \leq \infty$. For the case of the Hilbert transform and $m = 1$ see [B]. For kernels with condition (3.1) and $m = 1$, see [STa and STb]. Finally, the unweighted case was considered in [CH and CRW].

B. Let us assume that $k(x, y)$ is a kernel satisfying condition (2.3) and that

$$Kf(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y| > \varepsilon} k(x, y) f(y) dy,$$

defines a bounded linear operator from L^p into L^q , $1 < p < \lambda$, $q^{-1} = p^{-1} - \lambda^{-1}$. Then, if $a \in \text{BMO}(\nu)$,

$$(C_a^m)^*(f)(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y| > \varepsilon} k(x, y) (a(x) - a(y))^m f(y) dy \right|,$$

is a bounded linear operator from $L^p(\alpha^p)$ into $L^q(\beta^q)$ provided that α^r and β^r belong to $A(p/r, q/r)$, $1 \leq r < p < \lambda \leq \infty$, $q^{-1} = p^{-1} - \lambda^{-1}$, and $\alpha\beta^{-1} = \nu^m$. In fact let ϕ and ψ be C^∞ functions such that

$$\chi_{[2, \infty)} \leq \phi \leq \chi_{[1, \infty)} \quad \text{and} \quad \chi_{[1, 2]} \leq \psi \leq \chi_{[1/2, 3]}.$$

We consider the l^∞ -valued operators

$$\Phi(f)(x) = \left(\int k(x, y) \phi(|x-y|/\varepsilon) f(y) dy \right)_{\varepsilon > 0}$$

and

$$\Psi(f)(x) = \left(\int |k(x, y)| \psi(|x - y|/\varepsilon) f(y) dy \right)_{\varepsilon > 0}.$$

The kernels of Φ and Ψ satisfy conditions (2.3) and (W_3) of (2.4) respectively. By the vector valued theory of the Calderón-Zygmund operators, see [FRT and RT], these operators are bounded from L^p into $L^q_{l^\infty}$, $1 < p < \lambda \leq \infty$, $q^{-1} = p^{-1} - \lambda^{-1}$. Therefore, Φ satisfies the hypotheses of Theorem (2.1) and Ψ those of Theorem (2.4). Let $a(x)$ be in $BMO(\nu)$ and $\tilde{a}(x)$ in $BMO_{l^\infty}(\nu)$ defined as $\tilde{a}_\varepsilon(x) = a(x)$. Then

$$\Phi_a^m(f)(x) = \left(\int k(x, y) (a(x) - a(y))^m \phi(|x - y|/\varepsilon) f(y) dy \right)_\varepsilon$$

and

$$\Psi_a^m(f)(x) = \left(\int |k(x, y)| |a(x) - a(y)|^m \psi(|x - y|/\varepsilon) f(y) dy \right)_\varepsilon,$$

are bounded linear operators from $L^p(\alpha^p)$ into $L^q_{l^\infty}(\beta^q)$ for $1 \leq r < p < \lambda \leq \infty$, $q^{-1} = p^{-1} - \lambda^{-1}$, $\alpha \beta^{-1} = \nu^m$, α^r and β^r in $A(p/r, q/r)$. If we define

$$\hat{C}_a^m(f)(x) = \left(\int_{|x-y|>\varepsilon} k(x, y) (a(x) - a(y))^m f(y) dy \right)_{\varepsilon > 0},$$

then, the difference

$$\begin{aligned} U_a^m(f)(x) &= \Phi_a^m(f)(x) - \hat{C}_a^m(f)(x) \\ &= \left(\int k(x, y) (a(x) - a(y))^m \{ \phi(|x - y|/\varepsilon) - \chi_{[1, \infty)}(|x - y|/\varepsilon) \} f(y) dy \right)_{\varepsilon > 0}, \end{aligned}$$

satisfies

$$\begin{aligned} \|U_a^m f(x)\|_{l^\infty} &\leq \sup_{\varepsilon > 0} \left(\int |k(x, y)| |a(x) - a(y)|^m \psi(|x - y|/\varepsilon) |f(y)| dy \right) \\ &= \|\Psi_a^m(f)(x)\|_{l^\infty}. \end{aligned}$$

Therefore, $U_a^m(f)$ is a bounded operator and this shows that $\hat{C}_a^m(f)$ is bounded under the same conditions. Thus,

$$(C_a^m)^*(f)(x) = \|\hat{C}_a^m(f)(x)\|_{l^\infty},$$

is a bounded operator.

C. Let

$$Sf(x) = \left\{ \int \frac{e^{-iry}}{x - y} f(y) dy \right\}_{r \in \mathbf{R}}.$$

Since, by the Carleson-Hunt theorem, see [H], this operator is bounded from L^p into $L^p_{l^\infty}$, $1 < p < \infty$, and observing that the kernel $k(x, y) = e^{-iry}/(x - y)$ satisfies the conditions of Theorem (2.1) then

$$S_a^m(f) = \left\{ \int \frac{e^{-iry}}{x - y} (a(x) - a(y))^m f(y) dy \right\}_{r \in \mathbf{R}},$$

is a bounded operator from $L^p(\alpha^p)$ into $L^p_{l^\infty}(\beta^p)$, provided that $1 < p < \infty$, α and β belong to $A(p, p)$, and $\alpha \beta^{-1} = \nu^m$, and a belongs to BMO .

D. Let E be an U.M.D. space. Then, the Hilbert transform

$$Hf(x) = \text{p.v.} \int f(y)/(x-y) dy,$$

is known to be a bounded linear operator from L_E^p into L_E^p , see [Bu]. Therefore, by Theorem (2.1) we get that

$$C_a^m(f)(x) = \text{p.v.} \int (a(x) - a(y))^m f(y)/(x-y) dy,$$

is a bounded linear operator from $L_E^p(\alpha^p)$ into $L_E^p(\beta^p)$, α and β in $A(p, p)$, $\alpha\beta^{-1} = \nu^m$, $1 < p < \infty$, a belongs to $\text{BMO}_{\mathcal{L}(E, E)}$ and satisfies $a(x)a(y) = a(y)a(x)$ for every x and y .

E. **Maximal operators.** Let $0 \leq \phi \in L^1(\mathbf{R}^n)$ verify $|\phi(x-y) - \phi(x)| \leq c|y|/|x|^{n+1}$, provided that $|x| > 2|y|$. Set $\phi_\varepsilon(x) = \varepsilon^{-n/\lambda'} \phi(x/\varepsilon)$. Then the operator $M_\phi(f)(x) = (f * \phi_\varepsilon(x))_{\varepsilon>0}$, is a bounded linear operator from L^p into $L_{l^\infty}^q$, see [FRT and RT]. It is easy to check that the kernel $w(x, y) = \phi_\varepsilon(|x-y|)$ satisfies the hypotheses of Theorem (2.4). Therefore, if $a \in \text{BMO}(\nu)$, the operator

$$M_{\phi, a}^m(f)(x) = \left(\int |a(x) - a(y)|^m \phi_\varepsilon(x-y) f(y) dy \right)_{\varepsilon>0},$$

is a bounded operator from $L^p(\alpha^p)$ into $L_{l^\infty}^q(\beta^q)$, $1 < p < \lambda \leq \infty$, $q^{-1} = p^{-1} - \lambda^{-1}$, α and β in $A(p, q)$, $\alpha\beta^{-1} = \nu^m$, meaning that the operator

$$\sup_{\varepsilon>0} \int |a(x) - a(y)|^m \phi_\varepsilon(x-y) f(y) dy,$$

is bounded from $L^p(\alpha^p)$ into $L^q(\beta^q)$ under the same conditions on $p, q, \lambda, \alpha, \beta$ and ν . In particular taking ϕ such that $\chi_{[-1, 1]} \leq \phi$, we obtain that

$$S_a^m(f)(x) = \sup_{x \in Q} |Q|^{-1/\lambda'} \int_Q |a(x) - a(y)|^m f(y) dy,$$

is a bounded linear operator from $L^p(\alpha^p)$ into $L^q(\beta^q)$. Moreover we have the following theorem:

(3.2) **Theorem.** Let ν be such that $\nu^{m\lambda'/2} \in A(2, 2)$, then the following conditions are equivalent:

(a) For every p and q such that $q^{-1} = p^{-1} - \lambda^{-1}$, α and β in $A(p, q)$ and $\alpha\beta^{-1} = \nu^m$, the operator $S_a^m(f)$ maps $L^p(\alpha^p)$ into $L^q(\beta^q)$.

(b) The function $a(x)$ belongs to $\text{BMO}(\nu)$.

Proof. We have seen that (b) implies (a). Let us prove that (a) implies (b). By the factorization theorem of weights, see [RFC], there exists ν_0 and ν_1 such that $\nu^{m\lambda'} = \nu_0 \nu_1^{-1}$, ν_0 and ν_1 belonging to $A(1, 1)$. Let $p = 2\lambda/(\lambda+1)$, $q = 2\lambda'$, then $1 < p < \lambda$, $p' = q$, and $q^{-1} = p^{-1} - \lambda^{-1}$. Let $\alpha = (\nu_0 \nu_1^{-1})^{1/q}$ and $\beta = (\nu_0^{-1} \nu_1)^{1/q}$, then $\alpha\beta^{-1} = \nu^m$ and α and β belong to $A(p, q)$.

Therefore, we have

$$\begin{aligned}
 & \left(|Q|^{-1} \int_Q |a(x) - a_Q| dx \right)^m \leq |Q|^{-1} \int_Q |a(x) - a_Q|^m dx \\
 & \leq |Q|^{-2} \int_Q \left(\int_Q |a(x) - a(y)|^m dy \right) dx \\
 & \leq |Q|^{-1/p} \left(\int_Q (|Q|^{-1/\lambda'} \int_Q |a(x) - a(y)|^m dy)^q \beta^q(x) dx \right)^{1/q} \\
 & \quad \cdot \left(|Q|^{-1} \int_Q \beta^{-q'} dx \right)^{1/q'} \\
 & \leq |Q|^{-1/p} \left(\int_Q S_a^m(\chi_Q)^q \beta^q dx \right)^{1/q} \left(|Q|^{-1} \int_Q \beta^{-q'} dx \right)^{1/q'} \\
 & \leq c \left(|Q|^{-1} \int_Q \alpha^p dx \right)^{1/p} \left(|Q|^{-1} \int_Q \beta^{-q'} dx \right)^{1/q'}.
 \end{aligned}$$

Since $\alpha^p = \beta^{-q'} = \nu^{m\lambda/(\lambda+1)}$, $p = q' = 2\lambda/(\lambda+1)$ we get

$$\left(|Q|^{-1} \int_Q |a(x) - a_Q| dx \right)^m \leq c \left(|Q|^{-1} \int_Q \nu^m dx \right).$$

Then, by Lemma (4.4) in the following section, we obtain

$$|Q|^{-1} \int_Q |a(x) - a_Q| dx \leq c |Q|^{-1} \int_Q \nu(x) dx,$$

as claimed.

4. PROOFS OF THE COMMUTATOR THEOREMS

We begin this section with a sequence of lemmas.

(4.1) **Lemma.** *Let a belong to $\text{BMO}(\nu)$, $\alpha\beta^{-1} = \nu^m$, $m \geq 1$, α and β in $A(s', \infty)$, $1 \leq s < \infty$. Then*

$$\left(|Q|^{-1} \int_Q \|a(x) - a_Q\|^{ms} \alpha(x)^{-s} \right)^{1/s} \leq c \left(\inf_Q \beta^{-1} \right)$$

holds for every ball Q with a constant c depending on the class constants of α and β only.

For a proof of this lemma see [B].

(4.2) **Lemma.** *If $\nu^m \in A_\infty$ and satisfies*

$$|Q|^{-1} \int_Q \nu^{m(1+\varepsilon)}(x) dx \leq c \left(|Q|^{-1} \int_Q \nu^m(x) dx \right)^{1+\varepsilon},$$

for every ball Q ; c independent of Q , then

$$|Q|^{-1} \int_Q \nu(x)^{m(1+\varepsilon)} dx \leq c \left(|Q|^{-1} \int_Q \nu(x) dx \right)^{m(1+\varepsilon)},$$

for every ball Q , c independent of Q .

For a proof see [STW].

(4.4) **Lemma.** *If $a \in \text{BMO}(\nu)$ and (4.3) holds, then*

$$\left(|Q|^{-1} \int_Q \|a(x) - a_Q\|^{m(1+\varepsilon)} dx \right) \leq \left(|Q|^{-1} \int_Q \nu(x) dx \right)^{m(1+\varepsilon)}$$

holds for every ball Q with c independent of Q .

For a proof see [B].

(4.5) **Lemma.** *Let α and β be weights such that α^r and β^r belong to $A(((p/r)'t)', q)$, $t \geq 1$, and $\alpha\beta^{-1} = \nu^m$, $m \geq 1$. Then, if $\delta_k = \alpha^{(1-k/m)}\beta^{k/m}$, $0 \leq k \leq m$, we have*

$$\delta_k^t \in A(((p/r)'t)', q) \quad \text{and} \quad \delta_k \delta_h^{-1} = \nu^{h-k}, \quad \text{if } 0 \leq k < h \leq m.$$

Proof. We have

$$|Q|^{-1} \int_Q \delta^{-r(p/r)'t} dx = |Q|^{-1} \int_Q \alpha^{-r(p/r)'t(1-k/m)} \beta^{-r(p/r)'t k/m} dx.$$

Then, by Hölder inequality, we get that the integral above is smaller than or equal to

$$\left(|Q|^{-1} \int_Q \alpha^{-r(p/r)'t} dx \right)^{(1-k/m)} \left(|Q|^{-1} \int_Q \beta^{-r(p/r)'t} dx \right)^{k/m}.$$

Analogously,

$$|Q|^{-1} \int_Q \delta_k^q dx \leq \left(|Q|^{-1} \int_Q \alpha^q dx \right)^{(1-k/m)} \left(|Q|^{-1} \int_Q \beta^q dx \right)^{k/m},$$

and the lemma is proved.

(4.6) **Lemma.** *Let K be the operator in the statement of Theorem (2.1). Let Q be a ball and f a function with support contained in $4Q$. Then if α , β , and ν are weights such that $\alpha\beta^{-1} = \nu^m$, $m \geq 1$, α^r and β^r belong to $A(((\lambda/r)'v)', \infty)$, $v > 1$, we have*

$$\begin{aligned} & |Q|^{-1} \int_Q \|K((a - a_Q)^m f)(z)\| dx \\ (4.7) \quad & \leq c \left(\inf_Q \beta^{-1} \right) \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda}, \end{aligned}$$

where the constant c depends on the class constants of α and β only.

If W is the operator in the statement of Theorem (2.4), we have that

$$\begin{aligned} & |Q|^{-1} \int_Q \|W(|a - a_Q|^m |f|)(z)\| dz \\ (4.8) \quad & \leq c \left(\inf_Q \beta^{-1} \right) \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda}. \end{aligned}$$

holds with the same hypotheses as before.

Proof. Let p be defined by $p(\lambda/p)' = r(\lambda/r)'v$. Then $r < p < \lambda$ and if $1/q = 1/p - 1/\lambda$ we get

$$\begin{aligned}
 & |Q|^{-1} \int_Q \|K((a - a_Q)^m f)(z)\| dz \\
 & \leq |Q|^{-1/q} \left(\int_Q \|K((a - a_Q)^m f)(z)\|^q dz \right)^{1/q} \\
 & \leq c|Q|^{-1/q} \left(\int_{4Q} (\|a(y) - a_Q\|^m \|f(y)\|)^p dy \right)^{1/p} \\
 & \leq c|Q|^{-1/q} \left(\int_{4Q} \|a(y) - a_Q\|^{mp(\lambda/p)'} \alpha(y)^{-p(\lambda/p)'} dy \right)^{1/p(\lambda/p)'} \\
 & \quad \cdot \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda \right)^{1/\lambda}.
 \end{aligned}$$

Since $q = p(\lambda/p)'$ and applying Lemma (4.1) we obtain that (4.7) is bounded by $c(\inf_Q \beta^{-1}) \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda}$, as we wanted to show.

As for (4.8), the proof is the same.

(4.9) **Lemma.** Under the same hypotheses of Lemma (4.6), we have

$$\begin{aligned}
 (4.10) \quad & |Q|^{-1} \int_Q \|\tilde{a}(x) - \tilde{a}_Q\|^m \|Kf(x)\| dx \\
 & \leq c \left(\inf_Q \beta^{-1} \right) \left(\int \|f(x)\|^\lambda \alpha(x)^\lambda dx \right)^{1/\lambda},
 \end{aligned}$$

and

$$\begin{aligned}
 (4.11) \quad & |Q|^{-1} \int_Q |a(x) - a_Q|^m \|W(f)(x)\| dx \\
 & \leq \left(\inf_Q \beta^{-1} \right) \left(\int \|f(x)\|^\lambda \alpha(x)^\lambda dx \right)^{1/\lambda}.
 \end{aligned}$$

Proof. Let us choose ε , $0 < \varepsilon < 1$, such that

- (i) $r < \lambda(1 - \varepsilon)$.
- (ii) if $s = 1 + \varepsilon/((1 - \varepsilon)\lambda - \varepsilon)$, then $s < r(\lambda/r)'v$.
- (iii) $(\lambda(1 - \varepsilon)/r)' \leq (\lambda/r)'v$.

Then, let $p = \lambda(1 - \varepsilon)$. Then, by (i) we have $r < p < \lambda$ and $1/s' = 1/p - 1/\lambda$. Thus, the first member of (4.10) is less than or equal to

$$\left(|Q|^{-1} \int_Q \|\tilde{a}(x) - \tilde{a}_Q\|^{ms} \alpha(x)^{-s} dx \right)^{1/s} \left(|Q|^{-1} \int \|K(f)(x)\|^{s'} \alpha(x)^{s'} dx \right)^{1/s'}.$$

Since by (iii), $r(p/r)' \leq r(\lambda/r)'v$, we have

$$\begin{aligned}
 & \left(|Q|^{-1} \int_Q \alpha^{-r(p/r)'} dx \right)^{1/(p/r)'} \left(|Q|^{-1} \int_Q \alpha^q \right)^{1/(q/r)} \\
 & \leq c \left(\inf_Q \alpha^{-1} \right)^r \left(|Q|^{-1} \int_Q \alpha^q \right)^{r/q} \leq c,
 \end{aligned}$$

showing that α^r belong to $A(p/r, q/r)$. Thus

$$\begin{aligned} & \left(|Q|^{-1} \int \|K(f)(x)\|^{s'} \alpha(x)^{s'} dx \right)^{1/s'} \\ & \leq c |Q|^{-1/s'} \left(\int_{4Q} \|f(x)\|^p \alpha(x)^p dx \right)^{1/p} \\ & \leq c |Q|^{-1/s' + 1/p(\lambda/p)'} \left(\int \|f(x)\|^\lambda \alpha(x)^\lambda dx \right)^{1/\lambda} \\ & = c \left(\int \|f(x)\|^\lambda \alpha(x)^\lambda dx \right)^{1/\lambda}. \end{aligned}$$

Since by (ii) we know that $s < r(\lambda/r)'v$, then by Lemma (4.1)

$$\left(|Q|^{-1} \int_Q \|\tilde{a}(x) - \tilde{a}_Q\|^{ms} \alpha(x)^{-s} dx \right) \leq c \left(\inf_Q \beta^{-1} \right).$$

Therefore, (4.10) holds as claimed.

As for (4.11) the proof is the same. We observe that the lemma is true if $m = 0$. The proof is simple.

(4.12) **Lemma.** Let $a \in \text{BMO}(\nu)$, $\delta \beta^{-1} = \nu^k$, $k \geq 0$, δ and β belonging to $A(v', \infty)$, $v > 1$. Then

$$(4.13) \quad \left(|Q|^{-1} \int_Q |a - a_Q|^k dx \right) \left(|Q|^{-1} \int_Q |g| \right) \leq c \left(\inf_Q \beta^{-1} \right) \|\delta g\|_\infty$$

holds, with a constant c depending on the class constants of δ and β only.

Proof. We have

$$|Q|^{-1} \int_Q |g| dx \leq \|g\delta\|_\infty \left(|Q|^{-1} \int_Q \delta^{-1} dx \right) \leq c \|g\delta\|_\infty \left(\inf_Q \delta^{-1} \right).$$

Therefore, the first member of (4.13) is bounded by

$$c \left(|Q|^{-1} \int_Q |a - a_Q|^k \delta^{-1} \right) \|g\delta\|_\infty \leq c \left(|Q|^{-1} \int_Q |a - a_Q|^{kv} \delta^{-v} \right)^{1/v} \|g\delta\|_\infty,$$

and applying Lemma (4.1) we get (4.13).

(4.14) **Lemma.** Let $k(x, y)$ and $w(x, y)$ be kernels satisfying the hypotheses of Theorems (2.1) and (2.4). Assume that the support of $f(x)$ is contained in $C4Q$. Let α^r and β^r belong to $A(\lambda/r, \infty)$, $\alpha\beta^{-1} = \nu^m$, $m \geq 0$, a and $\tilde{a} \in \text{BMO}(\nu)$. Then,

if $z \in Q$, we have

$$\begin{aligned} (4.15) \quad & |Q|^{-1} \int_Q \|\tilde{a}(x) - \tilde{a}_Q\|^m \left(\int \|k(x, y) - k(z, y)\| \|f(y)\| dy \right) dx \\ & \leq c \left(\inf_Q \beta^{-1} \right) \left(\int \|f\|^\lambda \alpha^\lambda dx \right)^{1/\lambda}, \end{aligned}$$

if $x, z \in Q$, then

$$(4.16) \quad \begin{aligned} & \int \|k(x, y) - k(z, y)\| \|a(y) - a_Q\|^m \|f(y)\| dy \\ & \leq c \left(\inf_Q \beta^{-1} \right) \left(\int \|f\|^\lambda \alpha^\lambda dx \right)^{1/\lambda}, \end{aligned}$$

if $z \in Q$, we have

$$(4.15') \quad \begin{aligned} & |Q|^{-1} \int_Q |a(x) - a_Q|^m \left(\int \|w(x, y) - w(z, y)\| |f(y)| dy \right) dx \\ & \leq c \left(\inf_Q \beta^{-1} \right) \left(\int |f|^\lambda \alpha^\lambda dx \right)^{1/\lambda}, \quad \text{and} \end{aligned}$$

if $x, z \in Q$, then

$$(4.16') \quad \begin{aligned} & \int \|w(x, y) - w(z, y)\| |a(y) - a_Q|^m |f(y)| dy \\ & \leq c \left(\inf_Q \beta^{-1} \right) \left(\int |f|^\lambda \alpha^\lambda dx \right)^{1/\lambda} \end{aligned}$$

hold with a constant c depending on the class constants of α and β only provided that

$$\int_0^1 \lg^m(1/t) \omega_{r'}(t) dt/t < \infty.$$

Proof. We begin with (4.15). Let $Q = \{y: |y - x_0| < d\}$. We have

$$(4.17) \quad \begin{aligned} & \int \|k(x, y) - k(z, y)\| \|f(y)\| dy \\ & = \sum_{k=3}^{\infty} \int_{2^k Q \sim 2^{k-1} Q} \|k(x, y) - k(z, y)\| \|f(y)\| dy. \end{aligned}$$

Since for $R = 2^{k-1}d$ we have $2^k Q \sim 2^{k-1} Q = \{y: R \leq |y - x_0| < 2R\}$ and $|x - x_0| < d < R/2$, $|z - x_0| < d < R/2$, by Hölder inequality, and condition (2.3) we get that the integral under the summation sign is bounded by

$$\begin{aligned} & \left(\int_{2^k Q \sim 2^{k-1} Q} \|k(x, y) - k(z, y)\|^{r'} dy \right)^{1/r'} \left(\int_{2^k Q} \|f(y)\|^r dy \right)^{1/r} \\ & \leq c \left(2^{-k} + \int_{|x-z|/2^k d < t < |x-z|/2^{k-1}d} \omega_{r'}(t) dt/t \right) \\ & \quad \cdot \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda} \left(|2^k Q|^{-1} \int_{2^k Q} \alpha^{-r(\lambda/r)'} dy \right)^{1/r(\lambda/r)'}. \end{aligned}$$

Since $\alpha' \in A(\lambda/r, \infty)$, the second member of (4.17) is bounded by

$$c \left(\inf_Q \alpha^{-1} \right) \left(1 + \int_0^1 \omega_{r'}(t) dt/t \right) \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda}.$$

Then, by Lemma (4.1), the expression (4.17) is less than or equal to

$$\begin{aligned} & c|Q|^{-1} \int_Q \|\tilde{a}(x) - \tilde{a}_Q\|^m \alpha(x)^{-1} dx \\ & \leq c \left(|Q|^{-1} \int \|\tilde{a}(x) - \tilde{a}_Q\|^{mr(\lambda/r)'} \alpha(x)^{-r(\lambda/r)'} dx \right)^{1/r(\lambda/r)'} \|f\| \alpha \|_\lambda \\ & \leq c \left(\inf_Q \beta^{-1} \right) \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda}, \end{aligned}$$

which proves (4.15).

Let us prove (4.16). We have

$$\begin{aligned} & \int \|k(x, y) - k(z, y)\| \|a(y) - a_Q\|^m \|f(y)\| dy \\ & = \sum_{k=3}^{\infty} \int_{2^k Q \sim 2^{k-1} Q} \|k(x, y) - k(z, y)\| \|a(y) - a_Q\|^m \|f(y)\| dy. \end{aligned}$$

Since $\|a(y) - a_Q\| \leq \|a(y) - a_{2^k Q}\| + \|a_{2^k Q} - a_Q\|$ the integral under the summation sign is bounded by

$$\begin{aligned} & c \int_{2^k Q \sim 2^{k-1} Q} \|k(x, y) - k(z, y)\| \|a(y) - a_{2^k Q}\|^m \|f(y)\| dy \\ & + c \|a_{2^k Q} - a_Q\|^m \int_{2^k Q \sim 2^{k-1} Q} \|k(x, y) - k(z, y)\| \|f(y)\| dy = A_k + B_k. \end{aligned}$$

As for A_k , by Lemma (4.1) we get

$$\begin{aligned} A_k & \leq c \left(\int_{2^k Q \sim 2^{k-1} Q} \|k(x, y) - k(z, y)\|^{r'} dy \right)^{1/r'} \\ & \quad \cdot \left(\int_{2^k Q} \|a(y) - a_{2^k Q}\|^{mr(\lambda/r)'} \alpha(y)^{-r(\lambda/r)'} dy \right)^{1/r(\lambda/r)'} \\ & \quad \cdot \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda} c(2^k d)^{-1/r(\lambda/r)'} \\ & \leq \left(2^{-k} + \int_{|x-z|/2^k d < t < |x-z|/2^{k-1} d} \omega(t) dt/t \right) \\ & \quad \cdot \left(\inf_Q \beta^{-1} \right) \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda}. \end{aligned}$$

In order to estimate B_k , applying Hölder inequality twice, we obtain

$$\begin{aligned} B_k & \leq \|a_{2^k Q} - a_Q\|^m \left(\int_{2^k Q \sim 2^{k-1} Q} \|k(x, y) - k(z, y)\|^{r'} dy \right)^{1/r'} \\ & \quad \cdot \left(\int_{2^k Q} \alpha(x)^{r(\lambda/r)'} dx \right)^{1/r(\lambda/r)'} \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda}. \end{aligned}$$

Since $\|a_{2^k Q} - a_Q\| \leq \sum_{j=1}^k \|a_{2^j Q} - a_{2^{j-1} Q}\|$ and

$$\|a_{2^j Q} - a_{2^{j-1} Q}\| \leq 2^n |2^j Q|^{-1} \int_{2^j Q} \|a(y) - a_{2^j Q}\| dy,$$

we get

$$\begin{aligned}
 & \|a_2 k_Q - a_Q\|^m \left(\int_{2^k Q} \alpha(x)^{-r(\lambda/r)'} dx \right)^{1/r(\lambda/r)'} \\
 & \leq c \|a_2 k_Q - a_Q\|^m \left(\inf_{2^k Q} \alpha^{-1} \right) |2^k Q|^{1/r(\lambda/r)'} \\
 & \leq c k^m |2^k Q|^{1/r(\lambda/r)'} \max_{1 \leq j \leq k} \left(|2^j Q|^{-1} \int_{2^j Q} \|a(y) - a_{2^j Q}\|^{mr(\lambda/r)'} \alpha(y)^{-r(\lambda/r)'} dy \right)^{1/r(\lambda/r)'} \\
 & \leq c k^m \left(\inf_Q \beta^{-1} \right) |2^k Q|^{1/r(\lambda/r)'}.
 \end{aligned}$$

Therefore, by condition (2.3),

$$\begin{aligned}
 B_k & \leq c \left(\inf_Q \beta^{-1} \right) k^m \left(2^{-k} + \int_{|x-z|/2^k < t < |x-z|/2^{k-1}} \omega(t) dt/t \right) \\
 & \quad \cdot \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda}.
 \end{aligned}$$

Adding up our estimates on A_k and B_k , we get that (4.16) is bounded by

$$c \left(\inf_Q \beta^{-1} \right) \left(1 + \int_0^1 \lg^m(1/t) \omega_{r'}(t) dt/t \right) \left(\int \|f(y)\|^\lambda \alpha(y)^\lambda dy \right)^{1/\lambda},$$

as we wanted to show. The proofs of (4.15') and (4.16') are just the same.

Proofs of Theorems (2.1) and (2.4). Let us prove Theorem (2.4) first. Let us estimate $(W_m f)^*(x_0)$. Let Q be a ball and $x_0 \in Q$. Let us assume, as we may, that $a_Q = 0$. Let $f = f_1 + f_2$, where $f_1 = f \chi_{4Q}$. We define $c_Q \in F$ as

$$c_Q = |Q|^{-1} \int_Q \left(\int |a(y)|^m w(z, y) f_2(y) dy \right) dz.$$

Then,

$$\begin{aligned}
 & |Q|^{-1} \int_Q \|W_m f(x) - c_Q\| dx \\
 & \leq |Q|^{-1} \int_Q \|W_m f_1(x)\| dx + |Q|^{-1} \int_Q \|W_m f_2(x) - c_Q\| dx \\
 & = A_1 + A_2.
 \end{aligned}$$

For A_1 , we have

$$A_1 \leq c |Q|^{-1} \int_Q |a(x)|^m \|W(|f_1|)(x)\| dx + c |Q|^{-1} \int_Q \|W(|a|^m |f_1|)\| dx.$$

As for A_2 , recalling that for $v \in F$, $\|(|v|)\| = \|v\|$, we have

$$A_2 \leq |Q|^{-1} \int_Q \left\| |Q|^{-1} \int_Q \left(\int |a(y) - a(x)|^m w(x, y) - a(y)^m w(z, y) |f_2(y)| dy \right) dz \right\| dx.$$

Let us estimate the expression inside the norm. We have

$$\begin{aligned} & |Q|^{-1} \int_Q \left(\int |a(y) - a(x)|^m w(x, y) - a(y)^m w(z, y) |f_2(y)| dy \right) dz \\ & \leq \sum_{j=0}^{m-1} |a(x)|^{m-j} \int w(x, y) |a(y)|^j |f_2(y)| dy \\ & \quad + |Q|^{-1} \int_Q \left(\int |w(x, y) - w(z, y)| |a(y)|^m |f_2(y)| dy \right) dz = B_1 + B_2. \end{aligned}$$

Since $|a(y)|^i \leq |Q|^{-1} \int_Q |a(y) - a(z)|^i dz$, we get

$$\begin{aligned} B_1 & \leq c \sum_{i=0}^{m-1} |a(x)|^{m-j} |Q|^{-1} \int_Q \left(\int |a(y) - a(z)|^i |w(x, y) - w(z, y)| |f_2(y)| dy \right) dz \\ & \quad + \sum_{i=0}^{m-1} |a(x)|^{m-j} |Q|^{-1} \int_Q W_i(|f|)(z) dz = B_{11} + B_{12} \end{aligned}$$

and

$$B_{11} \leq c |Q|^{-1} \int_Q \int_Q (|a(x)|^m + |a(y)|^m + |a(z)|^m) |w(x, y) - w(z, y)| |f_2(y)| dy dz.$$

Thus, collecting estimates, and dropping the assumption $a_Q = 0$, we obtain that

$$\begin{aligned} & |Q|^{-1} \int_Q \|W_m f(x) - c_Q\| dx \\ & \leq c \left\{ |Q|^{-1} \int_Q |a(x) - a_Q|^m \|W(|f_1|)(x)\| dx + |Q|^{-1} \right. \\ & \quad \cdot \int_Q \|W(|a - a_Q|^m |f_1|)(x)\| dx \\ & \quad + |Q|^{-1} \int_Q |a(x) - a_Q|^m \int \|w(x, y) - w(z, y)\| |f_2(y)| dy \\ & \quad \left. + |Q|^{-1} \int_Q \int \|w(x, y) - w(z, y)\| |a(y) - a_Q|^m |f_2(y)| dy \right\} \\ & \quad + c \sum_{i=0}^{m-1} \left(|Q|^{-1} \int_Q |a(x) - a_Q|^{m-i} dx \right) \left(|Q|^{-1} \int_Q W_i(|f|)(z) dz \right) \\ & = c R_m(f, Q) + c \sum_{i=0}^{m-1} S_{m,i}(W_i(|f|), Q), \end{aligned}$$

where $R_m(f, Q)$ is the expression inside the brackets and

$$S_{m,i}(g, Q) = \left(|Q|^{-1} \int_Q |a(x) - a_Q|^{m-i} dx \right) \left(|Q|^{-1} \int_Q |g(z)| dz \right).$$

If we define $R_m(f)(x_0) = \sup_{x_0 \in Q} R_m(f, Q)$ and

$$S_{m,i}(g)(x_0) = \sup_{x_0 \in Q} S_{m,i}(g, Q),$$

we get

$$(4.18) \quad W_m(f)^\#(x_0) \leq c \left[R_m(f)(x_0) + \sum_{i=0}^{m-1} S_{m,i}(W_i(|f|))(x_0) \right].$$

Thus, by Lemmas (4.6), (4.9) and (4.14), we get

$$\|\beta R_m f\|_\infty \leq c \left(\int |f(x)|^\lambda \alpha(x)^\lambda dx \right)^{1/\lambda},$$

provided that $\nu^m = \alpha\beta^{-1}$, α^r and β^r belonging to $A(((\lambda/r)'v)', \infty)$, v any number greater than one and the constant c depending only on the class constants of α and β .

Also, by Lemma (4.12) we get $\|\beta S_{m,i}(g)\|_\infty \leq c\|\alpha g\|_\infty$, provided that $\alpha\beta^{-1} = \nu^{m-i}$, α and β belonging to $A(v', \infty)$, v any number greater than one and c depending on the class constants of α and β .

Let $T_0(f) = R_m(f\nu^{-m})$. Then

$$\begin{aligned} \|\beta T_0 f\|_\infty &= \|\beta R_m(f\nu^{-m})\|_\infty \leq c \left(\int |f|^\lambda \nu^{-m\lambda} \alpha^\lambda dx \right)^{1/\lambda} \\ &= c \left(\int |f|^\lambda \beta^\lambda dx \right)^{1/\lambda} \end{aligned}$$

whenever β^r and $(\nu^m \beta)^r$ belong to $A(((\lambda/r)'v)', \infty)$.

By the Extrapolation Theorem (1.3) we get that $\|\beta T_0 f\|_q \leq c\|\beta f\|_p$ holds for β^r and $(\nu^m \beta)^r$ belonging to $A(((p/r)'t)', q)$, $q^{-1} = p^{-1} - \lambda^{-1}$. Thus $\|\beta R_m(f)\|_q \leq c\|\alpha f\|_p$, provided $\alpha\beta^{-1} = \nu^m$.

Let $T_i(g) = S_{m,i}(g\nu^{i-m})$. Then by Lemma (4.12), we have

$$\|\beta T_i g\|_\infty = \|\beta S_{m,i}(g\nu^{i-m})\|_\infty \leq c\|g\nu^{i-m}\|_\infty = c\|g\beta\|_\infty$$

whenever β and $\delta = \nu^{m-i}\beta$ belong to $A(v', \infty)$. Then by the Extrapolation Theorem (1.3), in the case $\lambda = \infty$, we get $\|\beta T_i g\|_q \leq c\|g\beta\|_q$ for every β such that β^r and $(\nu^{m-i}\beta)^r$ belong to $A(((q/r)'t)', q/r)$. Therefore, if δ_i is the weight given in Lemma (4.5) and observing that since $p \leq q$, $A(((q/r)'t)', q/r) \subset A(((p/r)'t)', q/r)$, we have that

$$\begin{aligned} \|\beta S_{m,i}(W_i f)\|_q &= \|\beta T_i(\nu^{m-i} W_i(f))\|_q \\ &\leq c\|\beta \nu^{m-i} W_i(f)\|_q = c\|\delta_i W_i(f)\|_q \leq c\|\alpha f\|_p \end{aligned}$$

holds. Thus, by (4.18), we obtain

$$\|\beta(W_m f)^\#\|_q \leq c\|\beta R_m(f)\|_q + c \sum_{i=0}^{m-1} \|\beta S_{m,i}(W_i f)\|_q \leq c\|\alpha f\|_p,$$

as we wanted to show.

In order to finish the proof of Theorem (2.4) we have to show that the sharp function theorem, see [CF and RFC], applies. Let $f \in C_0^\infty$ and let Q be any ball containing the support of f . Then, the function $W_m f(x)$ is well defined and belongs to $L^{(1+\varepsilon)}$ on Q . In fact,

$$\|W_m f(x)\| \leq c|a(x) - a_Q|^m \|W(|f|)(x)\| + c\|W(|a - a_Q||f|)(x)\|.$$

Then,

$$\begin{aligned}
 (4.19) \quad & \int_Q \|W_m f(x)\|^{1+\varepsilon} dx \leq c \left(\int_Q |a - a_Q|^{m(1+\varepsilon)s} dx \right)^{1/s} \\
 & \cdot \left(\int_Q \|W(|f|)(x)\|^{(1+\varepsilon)s'} dx \right)^{1/s'} \\
 & + |Q|^{1/s} \left(\int_Q \|W(|a - a_Q|^m |f|)(x)\|^{(1+\varepsilon)s'} dx \right)^{1/s'}.
 \end{aligned}$$

If s and ε are chosen so that $(1 + \varepsilon)s$ is close enough to zero, then, by Lemma (4.4)

$$\left(\int_Q |a - a_Q|^{m(1+\varepsilon)s} dx \right)^{1/s} < \infty.$$

Moreover, since $(1 + \varepsilon)s'$ can be assumed to be greater than λ' , by the boundedness of the operator W and Lemma (4.4) again, the second member of (4.19) is seen to be finite. Let M_Q be the Hardy-Littlewood maximal function restricted to cubes contained in Q and analogously f^{*Q} the sharp function obtained by using the same cubes. Then, if $0 < \eta$ is small enough we have

$$\begin{aligned}
 (4.20) \quad & \int_Q \beta^q M_Q(\|Wf\|)^\eta dx \leq \left(\int_Q M_Q(\|Wf\|)^{1+\varepsilon} dx \right)^{\eta/(1+\varepsilon)} \\
 & \cdot \left(\int_Q \beta^{q((1+\varepsilon)/\eta)'} dx \right)^{1/((1+\varepsilon)/\eta)'}.
 \end{aligned}$$

Since η is as small as we please, $[(1 + \varepsilon)/\eta]'$ is sufficiently close to one, therefore, by the maximal theorem we see that the second member of (4.20) is finite. Then, by the sharp function theorem we have

$$\begin{aligned}
 \int_Q \|Wf\|^q \beta^q dx & \leq c \int_Q (\|Wf\|^{*,Q} \beta)^q dx \\
 & \leq c \int_Q ((Wf)^\# \beta)^q dx \leq c \int_Q ((Wf)^\# \beta)^q dx,
 \end{aligned}$$

where the constant depends on β only. Thus, enlarging Q we get

$$\left(\int_Q \|Wf\|^q \beta^q dx \right)^{1/q} \leq c \left(\int_Q ((Wf)^\#)^q \beta^q dx \right)^{1/q} \leq c \left(\int |f|^p \alpha^p dx \right)^{1/p},$$

as we wanted to show.

Proof of Theorem (2.1). Let Q be a ball. By conditions (i) and (iii) of Theorem (2.1), it follows easily by induction on m that

$$(4.21) \quad C^m(a_Q f) = \tilde{a}_Q C^m(f), \quad C^m_{a, \tilde{a}}(f) = C^m_{a-Q, \tilde{a}-\tilde{a}_Q}(f),$$

and

$$(4.22) \quad C^m(f)(x) = \sum_{k=0}^m (-1)^k \binom{k}{m} \tilde{a}(x)^{m-k} K(a^k f)(x).$$

By (4.21), we can assume that $a_Q = 0$. By elimination, from (4.22) we get that for suitable constants r_k , we have

$$C^m(f) = \sum_0^{m-1} r_k \tilde{a}^{m-k} C^k f + (-1)^m K(a^m f).$$

Let $f = f_1 + f_2$, $f_1 = f \cdot \chi_{4Q}$. We define c_Q as

$$\begin{aligned} c_Q &= (-1)^m |Q|^{-1} \int_Q K(a^m f_2)(z) dz \\ &= (-1)^m |Q|^{-1} \int_Q \left(\int k(z, y) a(y)^m f_2(y) dy \right) dz. \end{aligned}$$

The vector c_Q is well defined if f has a compact support. Then,

$$\begin{aligned} C^m(f)(x) - c_Q &= \left(\sum_{k=0}^{m-1} r_k \tilde{a}(x)^{m-k} C^k(f)(x) \right) + (-1)^m K(a^m f_1)(x) \\ &\quad + ((-1)^m K(a^m f_2)(x) - c_Q). \end{aligned}$$

Since, for $x \in Q$, we have

$$\begin{aligned} &(-1)^m K(a^m f_2)(x) - (-1)^m |Q|^{-1} \int_Q K(a^m f_2)(z) dz \\ &= (-1)^m |Q|^{-1} \int_Q \int [k(x, y) - k(z, y)] a^m(y) f_2(y) dy dz, \end{aligned}$$

we get (dropping the assumption $a_Q = 0$)

$$\begin{aligned} &|Q|^{-1} \int_Q \|C^m f(x) - c_Q\| dx \\ &\leq c \sum_{k=0}^{m-1} |Q|^{-1} \int_Q \|\tilde{a}(x) - \tilde{a}_Q\|^{m-k} \|C^k f(x)\| dx \\ &\quad + c \left\{ |Q|^{-1} \int_Q \|K((a - a_Q)^m f_1)(x)\| dx \right. \\ &\quad \left. + |Q|^{-1} \int_Q \left(\int \|k(x, y) - k(z, y)\| \|a(x) - a_Q\|^m \|f_2(y)\| dy \right) dz \right\} \\ &= c \left\{ \sum_{k=0}^{m-1} R_k(C^k f, Q) + S(f, Q) \right\}, \end{aligned}$$

where $R_k(g, Q) = |Q|^{-1} \int_Q \|\tilde{a}(x) - a_Q\|^{m-k} \|g\| dx$.

Thus, if $R_k(g)(x_0) = \sup_{x_0 \in Q} R_k(g, Q)$ and $S(f)(x_0) = \sup_{x_0 \in Q} S(f, Q)$, we obtain the estimate

$$C^m(f)^\#(x_0) \leq c \left\{ \sum_{k=0}^{m-1} R_k(C^k f)(x_0) + S(f)(x_0) \right\}.$$

From this estimate, the proof of the theorem is almost the same as that of Theorem (2.4) and will not be given.

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